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The phase structure of mass-deformed $\mathrm{SU}(2) \times \mathrm{SU}(2)$ quiver theory

Timothy J. Hollowood and Tom Kingaby

Department of Physics, University of Wales Swansea
Swansea, SA2 8PP, UK
E-mail: t.hollowood@swan.ac.uk, pytk@swan.ac.uk

Abstract: The phase structure of the finite $\mathrm{SU}(2) \times \mathrm{SU}(2)$ theory with $\mathcal{N} = 2$ supersymmetry, broken to $\mathcal{N} = 1$ by mass terms for the adjoint-valued chiral multiplets, is determined exactly by compactifying the theory on a circle of finite radius. The exact low-energy superpotential is constructed by identifying it as a linear combination of the hamiltonians of a certain symplectic reduction of the spin generalized elliptic Calogero-Moser integrable system. It is shown that the theory has four confining, two Higgs and two massless Coulomb vacua which agrees with a simple analysis of the tree-level superpotential of the four-dimensional theory. In each vacuum, we calculate all the condensates of the adjoint-valued scalars.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Supersymmetric Effective Theories.

Contents

1. Introduction

This paper is concerned with the problem of calculating the phase structure of certain finite $\mathcal{N} = 2$ theories perturbed by mass terms to $\mathcal{N} = 1$. The main paradigm for this is the $\mathcal{N} = 4$ theory perturbed by mass terms to $\mathcal{N} = 1$ (which one can think of passing through $\mathcal{N} = 2$ on the way): the so-called $\mathcal{N} = 1^*$ theory. Let us describe this case in more detail. Gauge theory with $\mathcal{N} = 4$ supersymmetry is a finite theory for which S-duality — actually $\mathrm{SL}(2, \mathbb{Z})$ — is an exact symmetry. However, once broken to $\mathcal{N} = 1$ by adding mass terms for the three adjoint chiral multiplets, the duality is broken: instead of being an exact symmetry it now relates different vacua of the theory [1, 2]. For instance, the weakly-coupled Higgs vacuum is related to one of the strongly-coupled confining vacua by $\tau \to -1/\tau$, where τ is the usual complexified coupling of the theory. More precisely, we know on the basis of semi-classical reasoning that, for gauge group $\mathrm{SU}(N)$, the vacua are associated to the partitions of N. Furthermore, those with a mass gap are associated to the subset of equipartitions: $N = p \cdot q$. In these vacua there is an unbroken $\mathrm{SU}(p)$ gauge symmetry and hence using standard reasoning based on the Witten Index there should be an additional degeneracy of p. Consequently the total number of massive vacua is equal to $\sum_{p|N} p$, a sum over the integer divisors of N. In particular, the Higgs vacua corresponds to $p = 1$ and the N confining vacua to $p = N$. These vacua form a finite-dimensional representation of $\mathrm{SL}(2, \mathbb{Z})$.

One way to investigate the vacuum structure of the mass-deformed theory is to realize the mass deformation in a two-stage process: first breaking to $\mathcal{N} = 2$ with a massive adjoint hypermultiplet. The Coulomb branch of the $\mathcal{N} = 2$ theory is described in the by-now standard way by a Seiberg-Witten curve [3]. Further breaking to $\mathcal{N} = 1$ can be understood as a perturbation which lifts most of the Coulomb branch to leave the vacua of the $\mathcal{N} = 1$ theory. In principle, the Seiberg-Witten curve can be used to find the vacua and all the condensates of lowest component chiral superfields in each vacuum. In particular, the massive vacua are associated to points on the Coulomb branch for which the associated Seiberg-Witten curve Σ undergoes maximal degeneration. Since the curve

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Σ is an N-sheeted cover of the underlying torus with complex structure τ [1] the maximal degeneration involves unbranched (unramified) N-fold covers of the torus. It is known in the $\mathcal{N}=4$ case that the Seiberg-Witten curve is the spectral curve of the elliptic Calogero-Moser integrable system [4].

An alternative and more direct approach for which the integrable system plays a central rôle involves compactifying the four-dimensional theory to three dimensions on a circle of finite radius [2, 5]. In three dimensions, the $\mathcal{N}=2$ theory has a Coulomb branch of twice the dimension of the four-dimensional theory due to the Wilson lines and dual photons of the unbroken abelian gauge group. What is particularly nice about this, is that the integrable dynamical system mentioned above now stands centre stage since the larger Coulomb branch is nothing but its (complexified) phase space. In contrast, the Coulomb branch of the four-dimensional theory only corresponds to the action variables alone. In other words, the Wilson lines and dual photons supply the missing angle variables. Further soft breaking to $\mathcal{N}=1$ is realized by adding a superpotential which is simply one of the action variables of the integrable system. The superpotential calculated in this way is actually exact, i.e. includes all the quantum corrections. One way to see this is to interpret the whole set-up in terms of the mirror map in three dimensions: in this case the integrable system arises as the Higgs branch of the mirror-dual theory in the form of a Hitchin system which, as such, is not subject to quantum corrections [5]. The superpotential is also independent of the compactification radius and so vacua and condensates extracted from it are also valid in the decoupling limit.

In this picture, the vacua of the theory are identified with the critical points of the exact superpotential and, since the latter is a hamiltonian of the integrable system, this means they are associated to equilibrium positions for the evolution, or flow, generated by that hamiltonian:

$$\text{Vacuum} \qquad \longleftrightarrow \qquad \text{equilibrium position of a given flow}$$

Massive vacua are special in that they are equilibrium positions for *any* choice of hamiltonian in the space of action variables:

$$\text{Massive Vacuum} \qquad \longleftrightarrow \qquad \text{equilibrium position for all the flows}$$

Formally this follows from the following line of reasoning. The angle variables take values in the jacobian of the spectral curve of the integrable system (in this case the Seiberg-Witten curve) Σ and upon maximal degeneration of Σ all the angle variables are frozen to a point and nothing moves. More concretely, this fact was also proved directly in the case of the $\mathcal{N}=1^*$ theory in the appendix of [6]. All the massive vacua have been found in this case by extremizing the superpotential. For the massless vacua the situation is not so well understood: although a complete picture is available for SU(3), for $N>3$ there are only partial results [2, 7].

In [8, 9] the whole picture described above was generalized to certain finite $\mathcal{N}=2$ theories, the so-called "quiver models", which arise from certain brane configurations in string theory. These theories have product $\mathrm{SU}(N)^k$ gauge symmetry. For these theories there is also an underlying integrable system arising as a Hitchin system which was identified as the spin-generalized elliptic Calogero-Moser system developed in refs. [10, 11, 12]. Once again the Coulomb branch of the compactified theory is identified with the phase space of the (complexified) integrable system and the exact superpotential describing the breaking to $\mathcal{N}=1$ is one of the hamiltonians.

A complete picture of all the massive vacua was found in [8] generalizing the situation in the $\mathcal{N}=1^*$ theory. However, as in the $\mathcal{N}=1^*$ theory, the situation with the massless vacua is not understood. This provides the motivation for the present work. In it we shall investigate the simplest quiver model with gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$. In this case, we will be able to find the complete vacuum structure explicitly including both the previously known massive but now also the massless vacua.

2. The phase structure via semi-classical reasoning

In this section we shall infer the phase structure of the theory by investigating the tree-level superpotential.

The $\mathrm{SU}(2)\times\mathrm{SU}(2)$ $\mathcal{N}=2$ supersymmetric quiver theory is an example of the more general $\mathrm{SU}(N)^k$ theories which we now define. The field content consists of (i) for each $\mathrm{SU}(N)$ factor an $\mathcal{N}=1$ vector multiplet and adjoint-valued chiral multiplet Φ_i, $i=1,\ldots,k$, and (ii) chiral multiplets Q_i, $\tilde{Q}_i$, $i=1,\ldots,k$, in the $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$ of $\mathrm{SU}(N)_i\times\mathrm{SU}(N)_{i+1}$, respectively. The tree-level superpotential, including the mass-deformation to $\mathcal{N}=1$, has the form

$$W=\frac{1}{g^2}\,\mathrm{Tr}\left\{\Phi_iQ_i\tilde{Q}_i-Q_i\Phi_{i+1}\tilde{Q}_i+m_iQ_i\tilde{Q}_i+\mu_i\Phi_i^2\right\}, \tag{2.1}$$

where we assume that the labels are defined modulo N. Here, m_i are the $\mathcal{N}=2$ supersymmetry preserving masses of the hypermultiplets and μ_i are the $\mathcal{N}=2\to\mathcal{N}=1$ breaking masses of the adjoint chiral multiplets.

We can investigate the vacuum structure of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ theory by solving the F-flatness conditions modulo complex gauge transformations in the usual way. The analysis was done for the massive vacua in the more general setting of the $\mathrm{SU}(N)^k$ theory in [8] and we can simply quote the results in this case. The solutions for the massless vacua are new.

First of all, there are confining vacua for which $\Phi_i=Q_i=\tilde{Q}_i=0$ and the gauge symmetry is completely unbroken. We expect that the theory at low energy is pure $\mathcal{N}=1$ Yang-Mills with gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Since each $\mathrm{SU}(2)$ factor is independent and each on its own yields two independent vacua, in all we expect four confining vacua.

There are two Higgs vacua in which the gauge group is completely broken. For the first

$$\begin{aligned}
&\Phi_1=\frac{1}{2}(m_1+m_2)\,\mathrm{diag}\,\big(1,-1\big)\,, &&\Phi_2=\frac{1}{2}(m_1-m_2)\,\mathrm{diag}\,\big(1,-1\big)\,,\\
&Q_1=\begin{pmatrix}0&0\\1&0\end{pmatrix}, &&\tilde{Q}_1=\begin{pmatrix}0&m_1(\mu_1+\mu_2)+m_2(\mu_1-\mu_2)\\0&0\end{pmatrix},\\
&Q_2=\begin{pmatrix}1&0\\0&0\end{pmatrix}, &&\tilde{Q}_2=\begin{pmatrix}m_1(\mu_1-\mu_2)+m_2(\mu_1+\mu_2)&0\\0&0\end{pmatrix}.
\end{aligned}\tag{2.2}$$

The other Higgs vacuum is obtained by swapping $\Phi_1 \leftrightarrow \Phi_2$ along with

$$Q_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{Q}_1 = \begin{pmatrix} 0 & m_1(\mu_1 - \mu_2) + m_2(\mu_1 + \mu_2) \\ 0 & 0 \end{pmatrix},$$
$$Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{Q}_2 = \begin{pmatrix} 0 & 0 \\ 0 & m_1(-\mu_1 + \mu_2) + m_2(\mu_1 + \mu_2) \end{pmatrix}. \tag{2.3}$$

In total, therefore, there are 6 vacua with a mass gap: 4 confining and 2 Higgs.

There are two massless, or Coulomb, vacua each with an unbroken U(1) factor. For the first

$$\Phi_1 = \frac{\mu_2 m_2}{\mu_1 + \mu_2}\,\mathrm{diag}\,(1, -1)\,, \qquad \Phi_2 = \frac{\mu_1 m_2}{\mu_1 + \mu_2}\,\mathrm{diag}\,(-1, 1)\,,$$
$$Q_1 = \tilde{Q}_1 = 0\,, \qquad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{Q}_2 = \frac{4m_2\mu_1\mu_2}{\mu_1 + \mu_2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.4}$$

whilst for the second

$$\Phi_1 = \frac{\mu_2 m_1}{\mu_1 + \mu_2}\,\mathrm{diag}\,(1, -1)\,, \qquad \Phi_2 = \frac{\mu_1 m_1}{\mu_1 + \mu_2}\,\mathrm{diag}\,(1, -1)\,,$$
$$Q_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{Q}_1 = \frac{4m_1\mu_1\mu_2}{\mu_1 + \mu_2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Q_2 = \tilde{Q}_2 = 0\,. \tag{2.5}$$

The analysis above holds for generic values of the masses. However, for particular values of the masses flat directions emerge and different vacua can be related. Of course at this stage we emphasize that we are not taking account any of the quantum effects. To start with, if m_1 or m_2 vanish then the two Higgs vacua are related by a flat direction. For instance with $m_2 = 0$ we have

$$\Phi_1 = \Phi_2 = \frac{1}{2}m_1\,\mathrm{diag}\,(1, -1)\,, \qquad Q_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$\tilde{Q}_1 = \begin{pmatrix} 0 & m_1(\mu_1 + \mu_2) \\ 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{Q}_2 = \begin{pmatrix} x & 0 \\ 0 & x - m_1(\mu_1 - \mu_2) \end{pmatrix}. \tag{2.6}$$

Here, x parameterizes the flat direction. In an analogous way, one of the massless vacua is related to the confining vacuum by a flat direction.

3. The exact superpotential

Having established in the last section a qualitative picture based on the tree-level superpotential of the theory in four dimensions, we can now investigate the phase structure exactly following ref. [8]. As we alluded to in the introduction, the superpotential is precisely one of the hamiltonians in the space of action variables of a complexified integrable system. For our theory the latter is a certain symplectic reduction of the spin generalization of the elliptic Calogero-Moser system. We now construct it for the general $\mathrm{SU}(N)^k$ theory. In

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this case it describes the motion of N particles in one dimension with positions X_a and momenta p_a. Each particle carries a "spin" in the form of a $k \times k$ matrix with elements $\mathfrak{J}^a_{ij}$. The basic hamiltonian of the system is[1]

$$
\begin{aligned}
H_0 = \sum_a p_a^2 + \sum_{a \neq b} \sum_{ij} \mathfrak{J}^a_{ji} \mathfrak{J}^b_{ij} \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})} \big(\zeta(X_{ab} + z_{ji}) - \zeta(X_{ab})\big) - \\
- \frac{1}{2} \sum_{i \neq j} \Big[\sum_a \mathfrak{J}^a_{ij} \mathfrak{J}^a_{ji} - N m_i m_j \Big] \Big(\wp(z_{ij}) - \zeta(z_{ij})^2 \Big) .
\end{aligned}
\tag{3.1}
$$

Here, $\wp(z)$ is the Weierstrass function and $\sigma(z)$ and $\zeta(z)$ are its cousins defined on the torus with half-periods $\omega_1 = i\pi$ and $\omega_2 = i\pi\tau$ (so of complex structure τ) [13]. In the above, the separation between the particles is given by $X_{ab} \equiv X_a - X_b$ while $z_{ij} \equiv z_i - z_j$ are "inhomogeneities", $k-1$ external parameters (since only the differences matter). In our application, the k independent complex coupling constants τ_i of each of the $\mathrm{SU}(N)$ factors of the gauge group are associated to the k independent parameters $\{\tau, z_i\}$ in the following way. Firstly we order the z_i so that $0 \leq \mathrm{Re}\, z_i \leq \mathrm{Re}\, z_{i+1} \leq 2\pi \,\mathrm{Im}\, \tau$. Then

$$
\tau_i = i \frac{z_{i+1} - z_i}{2\pi} \qquad i = 1, \ldots, k-1 , \qquad \tau_k = i \frac{z_1 - z_k}{2\pi} + \tau . \tag{3.2}
$$

To define the dynamical system the dynamical variables have the non-vanishing Poisson brackets [11]

$$
\{X_a, p_b\} = \delta_{ab} , \qquad \{\mathfrak{J}^a_{ij}, \mathfrak{J}^b_{kl}\} = \delta_{ab} \big(\delta_{jk} \mathfrak{J}^a_{il} - \delta_{il} \mathfrak{J}^a_{kj} \big) . \tag{3.3}
$$

In fact, in the application to gauge theory, the spins are not arbitrary $k \times k$ matrices, rather they have rank one and so we can define them in terms of new variables Q_{ai} and $\tilde{Q}_{ia}$:

$$
\mathfrak{J}^a_{ij} = \tilde{Q}_{ia} Q_{aj} . \tag{3.4}
$$

If we take all the inhomogeneities z_i equal, then (3.1) simplifies to

$$
H_0 = \sum_a p_a^2 - \sum_{a \neq b} \mathrm{Tr} \left(\mathfrak{J}^a \mathfrak{J}^b \right) \wp(X_{ab}) , \tag{3.5}
$$

the dynamical system analysed in ref. [11]. The system is completely integrable, even when the z_i are arbitrary, so there exists a basis of action-angle variables for which the hamiltonian (3.1) is but one of a set of action variables.

For the application to gauge theory, we have to impose additional conditions on the spins. The reduction can be defined as a symplectic quotient by the abelian symmetries

$$
Q_{ai} \to e^{\phi_a} Q_{ai} e^{\psi_i} , \qquad \tilde{Q}_{ia} \to e^{-\psi_i} \tilde{Q}_{ia} e^{-\phi_a} . \tag{3.6}
$$

In all there are $N + k - 1$ independent symmetries. Taking the symplectic quotient involves imposing the momentum map constraints:

$$
\sum_a Q_{ai} \tilde{Q}_{ia} = N m_i , \qquad \sum_i Q_{ai} \tilde{Q}_{ia} = 0 , \tag{3.7}
$$

along with an ordinary quotient by the symmetries (3.6). Notice that hypermultiplet

[1] We have written the following in terms of spins $\mathfrak{J}^a_{ij}$ which is slightly different but equivalent to the way the system was written in [8] in terms of the spins $\mathbb{S}^i_{ab}$. The relation between the two representations can be determined from eq. (3.4).

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masses m_i enter via (3.7). In (3.1) we note that the centre-of-mass motion is completely trivial and so we set $\sum_a p_a = \sum_a X_a = 0$. Once this has been done the phase space (after the symplectic quotient of the spins) has the dimension $2k(N-1)$: precisely the complex dimension of the Coulomb branch of the compactified $\mathrm{SU}(N)^k$ theory. So the complexified phase is identified with the Coulomb branch of the compactified theory. This space is actually a hyper-Kähler manifold with a chosen complex structure. This is clear in the formulation of the integrable system as a Hitchin system [5, 8, 14] which has the form of an infinite-dimensional hyper-Kähler quotient [15]. In this context, the symplectic form of the dynamical system is identified with the closed $(2,0)$ form with respect to the chosen complex structure.

The remaining action variables can be extracted from the Lax operator described in [8]. Of particular importance for us is the basic hamiltonian (3.1) along with the following others

$$H_i = 2\sum_a p_a \mathfrak{J}^a_{ii} - 2\sum_{a\neq b}\sum_{j(\neq i)} \mathfrak{J}^a_{ji}\mathfrak{J}^b_{ij}\frac{\sigma(X_{ab}+z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})} + 2\sum_{j(\neq i)}\left[\sum_a \mathfrak{J}^a_{ij}\mathfrak{J}^a_{ji} - N m_i m_j\right]\zeta(z_{ij})\,. \tag{3.8}$$

of which only $k-1$ are independent since $\sum_{i=1}^k H_i = 0$.

The action variables, or hamiltonians, parameterize the Coulomb branch of the four-dimensional theory prior to compactification. In particular, the k independent hamiltonians H_0 along with H_i are identified with the subspace of quadratic condensates $\mathrm{Tr}\,\Phi_i^2$. In [8], we identified the unique combination of hamiltonians corresponding to the diagonal combination:

$$\sum_{i=1}^k \mathrm{Tr}\,\Phi_i^2 = k\,H^*\,, \tag{3.9}$$

where

$$H^* = H_0 - \frac{1}{k}\sum_{i\neq l}\zeta(z_{il})H_i\,. \tag{3.10}$$

The fact that there is a non-trivial function multiplying the H_i is required in order that H^* has the appropriate modular properties. The superpotential in the three-dimensional compactification corresponding to an arbitrary $\mathcal{N}=1$ mass deformation of the theory is then identified with a particular linear combination of the hamiltonians:

$$W = \frac{1}{g^2}\sum_{i=1}^k \mu_i\,\mathrm{Tr}\,\Phi_i^2 = \frac{1}{g^2}\left(\lambda_0(\mu)H^* + \sum_{i=1}^k \lambda_i(\mu)H_i\right)\,. \tag{3.11}$$

for quantities $\{\lambda_0, \lambda_i\}$ depending linearly on the μ_i.

In the case of an $\mathrm{SU}(2)\times\mathrm{SU}(2)$ quiver, we can be more explicit. Firstly regarding the symplectic reduction on the spins. Solving the moment map conditions (3.7) and fixing the symmetries (3.6) can be achieved, for instance, by parameterizing them with two variables $\{x,y\}$ such that

$$Q_{ai} = \begin{pmatrix} 1 & m_2 - y \\ e^{-x}\frac{m_1-y}{y-m_2} & m_2+y \end{pmatrix}\,, \qquad \tilde{Q}_{ia} = \begin{pmatrix} y+m_1 & e^x(y-m_2) \\ 1 & 1 \end{pmatrix}\,. \tag{3.12}$$

The Poisson bracket that one derives from (3.3) is then simply $\{x,y\}=1$.

Once this has been done, the dynamical system has a four-dimensional phase space parameterized by $X \equiv X_1 - X_2$, $p = \frac{1}{2}(p_1 - p_2)$, x and y with non-trivial Poisson brackets

$$\{X, p\} = 1\,, \qquad \{x, y\} = 1\,. \tag{3.13}$$

The two hamiltonians (3.1) and (3.8) are

$$\begin{aligned} H_0 =& 2p^2 + 2(2y^2 - m_1^2 - m_2^2)\wp(X) - 2e^x(y^2 - m_2^2)\frac{\sigma(X-z)}{\sigma(X)\sigma(z)}\big(\zeta(X-z) - \zeta(X)\big) + \\ &+ 2e^{-x}(y^2 - m_1^2)\frac{\sigma(X+z)}{\sigma(X)\sigma(z)}\big(\zeta(X+z) - \zeta(X)\big) + 2y^2(\wp(z) - \zeta(z)^2)\,, \end{aligned} \tag{3.14a}$$

$$H_1 = 4py - 4y^2\zeta(z) + 2e^x(y^2 - m_2^2)\frac{\sigma(X-z)}{\sigma(X)\sigma(z)} + 2e^{-x}(y^2 - m_1^2)\frac{\sigma(X+z)}{\sigma(X)\sigma(z)}\,, \tag{3.14b}$$

where $z \equiv z_{12}$. It is straightforward to check that H_0 and H_1 Poisson-commute.

In the case of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ we can uniquely identify the relation between the gauge invariants operators $\mathrm{Tr}\,\Phi_i^2$, $i = 1, 2$ and the hamiltonians. Firstly, as in (3.10) there is a unique combination which has the required properties to be identified with the average combination:

$$\frac{1}{2}\,\mathrm{Tr}\left(\Phi_1^2 + \Phi_2^2\right) \equiv H^* = H_0 - \zeta(z)H_1\,. \tag{3.15}$$

whilst the quantity H_1 is identified with the difference

$$\mathrm{Tr}\left(\Phi_2^2 - \Phi_1^2\right) \equiv H_1\,. \tag{3.16}$$

It follows that

$$\mathrm{Tr}\,\Phi_1^2 = H^* - \frac{1}{2}H_1\,, \qquad \mathrm{Tr}\,\Phi_2^2 = H^* + \frac{1}{2}H_1\,. \tag{3.17}$$

We can now identify the general $\mathcal{N} = 1^*$ deformation of the superpotential with the following linear combination of the action variables:

$$W = \frac{1}{g^2}\big(\mu_1\,\mathrm{Tr}\,\Phi_1^2 + \mu_2\,\mathrm{Tr}_2\;\Phi_2^2\big) = \frac{1}{g^2}(\mu_1 + \mu_2)\tilde{H}\,, \tag{3.18}$$

where

$$\begin{aligned} \tilde{H} =& H^* + \frac{1}{2}\beta H_1 = 2p^2 + 4\alpha py + 2e^x(y^2 - m_2^2)\tilde{\phi}(X) + 2e^{-x}(y^2 - m_1^2)\phi(X) + \\ &+ 2(2y^2 - m_1^2 - m_2^2)\wp(X) + 2y^2(\wp(z) - \zeta(z)^2 - 2\alpha\zeta(z))\,, \end{aligned} \tag{3.19}$$

where we have defined the constant

$$\alpha = -\zeta(z) + \frac{1}{2}\beta\,, \qquad \beta = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \tag{3.20}$$

along with the functions

$$\begin{aligned} \phi(X) &= \frac{\sigma(X+z)}{\sigma(X)\sigma(z)}\left(\zeta(X+z) - \zeta(X) + \alpha\right), \\ \tilde{\phi}(X) &= \frac{\sigma(X-z)}{\sigma(X)\sigma(-z)}\left(\zeta(X-z) - \zeta(X) - \alpha\right). \end{aligned} \tag{3.21}$$

4. The exact phase structure

Supersymmetric vacua are obtained by extremizing the superpotential (3.18). One obtains the equations

$$\frac{\partial \tilde{H}}{\partial p} = 4p + 4\alpha y = 0\,, \tag{4.1a}$$

$$\frac{\partial \tilde{H}}{\partial y} = 4y\big\{2\wp(X) + \wp(z) - \zeta(z)^2 - 2\alpha\zeta(z) + e^x\tilde{\phi}(X) + e^{-x}\phi(X)\big\} + 4\alpha p = 0\,, \tag{4.1b}$$

$$\frac{\partial \tilde{H}}{\partial x} = 2e^x(y^2 - m_2^2)\tilde{\phi}(X) - 2e^{-x}(y^2 - m_1^2)\phi(X) = 0\,, \tag{4.1c}$$

$$\frac{\partial \tilde{H}}{\partial X} = 2(2y^2 - m_1^2 - m_2^2)\wp'(X) + 2e^x(y^2 - m_2^2)\tilde{\phi}'(X) + 2e^{-x}(y^2 - m_1^2)\phi'(X) = 0\,. \tag{4.1d}$$

We now begin by solving (4.1a) for p:

$$p = -\alpha y\,. \tag{4.2}$$

One branch of solutions is then obtained by solving (4.1b) with $y = 0$. It then follows that there are two solutions of (4.1c), which we label by $n_1 = 1, 2$, for which

$$e^x = (-1)^{n_1}\frac{m_1}{m_2}\sqrt{\frac{\phi(X)}{\tilde{\phi}(X)}}\,. \tag{4.3}$$

Using standard elliptic function identities, along with (4.2) and (4.3), the final equation (4.1d) can be recast in the form

$$\wp'(X)\Big(m_1^2 + (-1)^{n_1}m_1 m_2\gamma\big(\phi\tilde{\phi}\big)^{-1/2} + m_2^2\Big) = 0\,, \tag{4.4}$$

where we have defined the quantity

$$\gamma = 2\wp(X) + \wp(z) - \frac{\beta^2}{4}\,. \tag{4.5}$$

For later use, one can show, again using standard elliptic function identities, that

$$\phi(X)\tilde{\phi}(X) = \wp^2(X) + \frac{1}{4}\beta^2\wp(z) + \wp^2(z) + \wp(X)\wp(z) - \frac{1}{4}\beta^2\wp(X) + \frac{1}{2}\beta\wp'(z) - \frac{1}{4}g_2\,, \tag{4.6}$$

from which one deduces

$$\gamma^2 - 4\phi\tilde{\phi} = g_2 - 3\wp^2(z) - 2\beta\wp'(z) - \frac{3}{2}\beta^2\wp(z) + \frac{1}{16}\beta^4\,. \tag{4.7}$$

As a consequence the left-hand side is independent of X.

For generic masses the solution to (4.4) is $\wp'(X) = 0$, i.e. X is a half-period

$$X \in \{\omega_1,\, \omega_2,\, \omega_1 + \omega_2\}\,, \tag{4.8}$$

which we label $X_c = i\pi, i\pi\tau, i\pi(\tau + 1)$, $c = 1, 2, 3$.

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In order to assess whether these six vacua are massive or massless, we compute the Hessian:

$$\mathrm{Det}\left[\frac{\partial^2 \tilde{H}}{\partial x_i \partial x_j}\right] = \frac{1}{m_2^2 \phi\tilde{\phi}}\Big((-1)^{n_1} m_1 m_2 \gamma + (m_1^2+m_2^2)(\phi\tilde{\phi})^{1/2}\Big)\times$$
$$\times\Big(2\phi\tilde{\phi}\wp''(X)\Big[(-1)^{n_1} m_1 m_2 \gamma + (m_1^2+m_2^2)(\phi\tilde{\phi})^{1/2}\Big] -$$
$$- (-1)^{n_1} m_1 m_2 (\gamma^2 - 4\phi\tilde{\phi})\wp'(X)\Big)\Big|_{X=X_c}. \tag{4.9}$$

It can be shown that the above is generically non-zero so that all six vacua are massive. The values of the condensates in these six massive vacua are

$$\mathrm{Tr}\,\Phi_i^2 = -\,2(m_1^2+m_2^2)\wp(X) -$$
$$-\,4(-1)^{n_1}\frac{m_1 m_2}{(\phi\tilde{\phi})^{1/2}}\Big(\phi\tilde{\phi} + \frac{1}{2}(\beta - (-1)^i)\big(\beta(\wp(z)-\wp(X)) + \wp'(z)\big)\Big)\Big|_{X=X_c}. \tag{4.10}$$

These six vacua are precisely the vacua found in [8] for general k and N. It is tempting to identify them with the six massive vacua, two Higgs and four confining, that we found in section 2 and this turns out to be correct. In order to pin down the relation, consider the semi-classical expansion of the condensates in each of the vacua as described in [9]. The expansions we need can be deduced from the following expansions of the (quasi-)elliptic functions

$$\wp(X) = \frac{1}{12} + \frac{e^{-X}}{(1-e^{-X})^2} + \sum_{n=1}^{\infty}\left\{\frac{e^{-X}q^n}{(1-e^{-X}q^n)^2} + \frac{e^{X}q^n}{(1-e^{X}q^n)^2} - \frac{2q^n}{(1-q^n)^2}\right\}, \tag{4.11}$$

$$\sigma(X) = e^{\zeta(i\pi)X^2/(2i\pi)}(e^{X/2} - e^{-X/2})\prod_{n=1}^{\infty}\frac{1-q^n(e^X - e^{-X}) + q^{2n}}{(1-q^n)^2}, \tag{4.12}$$

$$\zeta(X) = X\frac{\zeta(i\pi)}{i\pi} + \frac{1}{2}\coth\Big(\frac{X}{2}\Big) - \sum_{n=1}^{\infty}\frac{q^n(e^X + e^{-X})}{1-q^n(e^X+e^{-X}) + q^{2n}}. \tag{4.13}$$

The condensates can be written in terms of the complex couplings of each gauge group factor:

$$q_1 = e^{2\pi i \tau_1} = e^{-z}, \qquad q_2 = e^{2\pi i \tau_2} = q e^{z}, \tag{4.14}$$

where $q = e^{2\pi i \tau}$. It is easy to see that the condensates have an expansion in terms of the quantities

$$e^{-X}q^n, \qquad e^{X}q^{n+1}, \qquad q^n, \qquad e^{-z}q^n, \qquad e^{z}q^{n+1}, \tag{4.15}$$

with $n = 0, 1, 2, \ldots$. Given the values for X in (4.8), it is clear that the vacua with $X = i\pi$ have an expansion in integer powers of q_1 and q_2. Hence, the two vacua with $X = i\pi$ are identified with the Higgs vacua in which the condensates have a conventional semi-classical instanton expansion in integer powers of q_1 and q_2. The vacua with $X = i\pi\tau$ or $i\pi(\tau+1)$ have an expansion which includes powers of the fractional instanton factor $q^{1/2}$. This is characteristic of a confining vacuum. Hence, we identify the four vacua with these values of X and $n_1 = 1, 2$ with the four confining vacua identified in section 2.

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Now we return to the equations for the vacua (4.1a)–(4.1d) and choose a different branch of solutions obtained by solving (4.1b) with

$$e^x = \frac{-\gamma + (-1)^{n_2}\sqrt{\gamma^2 - 4\phi\tilde{\phi}}}{2\tilde{\phi}}\,, \tag{4.16}$$

rather than $y = 0$. There are two solutions of this type labelled by $n_2 = 1, 2$. Then (4.1c) is solved for y giving

$$y = \sqrt{\frac{m_2^2 e^x \tilde{\phi} - m_1^2 e^{-x}\phi}{e^x\tilde{\phi} - e^{-x}\phi}}\,. \tag{4.17}$$

Choosing the opposite sign for y can be shown to lead to an equivalent solution due to the presence of discrete symmetries which we have hitherto ignored. In particular the values of the condensates will not depend on it.

The final equation (4.1d) becomes

$$(m_1^2 - m_2^2)\frac{\partial\sqrt{\gamma^2 - 4\phi\tilde{\phi}}}{\partial X} = 0 \tag{4.18}$$

which is identically zero for all values of X since the combination (4.7) is independent of X.

The two solutions are obviously massless vacua since each corresponds to a line of critical points parameterized by X. The values of the condensates in these two massless vacua are

$$\begin{aligned}\mathrm{Tr}\,\Phi_i^2 =& (m_1^2 + m_2^2)\left(\wp(z) - \frac{1}{2}(-1)^i\beta + \frac{1}{4}\beta^2\right) + \\ &+ (-1)^{n_2}\frac{m_1^2 - m_2^2}{\sqrt{\gamma^2 - 4\phi\tilde{\phi}}}\left(\gamma^2 - 4\phi\tilde{\phi} + \frac{1}{2}\beta(\beta - (-1)^i)\left(3\wp(z) + 2\wp'(z) - \frac{1}{4}\beta^2\right)\right).\end{aligned} \tag{4.19}$$

The discussion of the vacuum structure above has been established in the case where the masses $\{m_i\}$ and $\{\mu_i\}$ are generic. For special values the vacua can merge. First of all, if X equals a half period and y in (4.17) equals 0, which requires

$$m_2^2 e^x \tilde{\phi} - m_1^2 e^{-x}\phi = 0\,, \tag{4.20}$$

where x is given by (4.16), then a massless vacuum meets what was once one of the massive vacua. Solving these equations leads to a condition on the ratio of the hypermultiplet masses m_1/m_2. In this way either of the massless vacua can meet any of the 6 massive vacua at 12 special values for m_1/m_2:

$$\frac{m_1}{m_2} = (-1)^{n_1}\frac{-\gamma + (-1)^{n_2}\sqrt{\gamma^2 - 4\phi\tilde{\phi}}}{2\sqrt{\phi\tilde{\phi}}} \tag{4.21}$$

with $X = X_c$, $c = 1, 2, 3$. Finally the two massless vacua merge together when

$$\gamma^2 - 4\phi\tilde{\phi} = g_2 - 3\wp^2(z) - 2\beta\wp'(z) - \frac{3\beta^2\wp(z)}{2} + \frac{\beta^4}{16} = 0\,. \tag{4.22}$$

5. Discussion

We have calculated the exact phase structure and the condensates of the two adjoint-valued scalar fields in the mass deformed $SU(2) \times SU(2)$ finite quiver theory. The strategy involved compactifying the theory on a circle of finite radius so that the low-energy degrees-of-freedom are all scalar. However, the values calculated remain valid in the decompactification limit. In this way, we were able to show how the exact structure of vacua matches the one deduced from an analysis of the tree-level superpotential in the four-dimensional theory. It would be interesting to extend our analysis to the general $SU(N)^k$ quiver theories and also to consider the solution of these mass-deformed theory using the matrix model formalism developed by Dijkgraaf and Vafa [16] also applied to the $\mathcal{N} = 1^*$ theory in [6, 17].

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